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GEOMETRICAL PROPERTIES OF THE MULTIDIMENSIONAL NONLINEAR DIFFERENTIAL EQUATIONS AND THE FINSLER METRICS OF PHASE SPACES OF DYNAMICAL SYSTEMS

The multidimensional nonlinear differential equations arising from special conditions on curvature tensor of 3- and 4-dimensional manifolds are considered. The examples of the Finsler metrics, associated with nonlinear dynamical systems are constructed.

1. INTRODUCTION

In the first part of this article we describe some nonlinear differential equations arising as conditions on the curvature tensor of 3– and 4–dimensional manifolds in the following form:

$$R_{ijkl} = \lambda(g_{ik}\, g_{jl} - g_{il}\, g_{jk})\,, \tag{1}$$

$$R_{ik} = R^l_{ilk} = 0\,, \tag{2}$$

$$^3R = g^{ij}(\Gamma^m_{im,j} - \Gamma^m_{ij,m} + \Gamma^m_{li}\,\Gamma^l_{jm} - \Gamma^l_{ml}\Gamma^m_{ij}) = 0\,, \tag{3}$$

$$R_{ik,i} = [\Gamma_i, R_{ik}]\,, \tag{4}$$

where $R^j_{ilk} = \Gamma^j_{il,k} - \Gamma^j_{lk,i} + [\Gamma_i, \Gamma_k]^j_l$ is the Riemannian tensor of the space with connection Γ^i_{jk} and metric g_{ij}, ^3R is the scalar curvature of the 3–dimensional space.

In the second part we discuss some questions of the Cartan theory for equations:

$$y'' = f(x, y, y')\,, \tag{5}$$

geometrical properties of the space of the linear elements (x, y, y') and will construct examples of the Finsler metrics for the second order differential equation equivalent to the Lorentz system

$$\dot{x} = k(x - y)\,, \qquad \dot{y} = rx - y - zx\,, \qquad \dot{z} = xy - bz\,. \tag{6}$$

2. AROUND CONDITION $R_{ijkl} = \lambda(g_{ik}\, g_{jl} - g_{il}\, g_{jk})$

Let us consider this condition for the metric of 3–dimensional space:

$$ds^2 = A^2(x, y, z)dx^2 + B^2(x, y, z)dy^2 + C^2(x, y, z)dz^2\,. \tag{7}$$

Then one obtains the system of equations:

$$
\begin{aligned}
(8)\qquad A_{zy} &= (C_y/C)A_z + (B_z/B)A_y\,, \quad (B_z/C)_z + (C_y/B)_y + B_xC_x/A^2 = \lambda BC\,,\\
B_{zx} &= (C_x/C)B_z + (A_z/A)B_x\,, \quad (C_x/A)_x + (A_z/C)_z + A_yC_y/B^2 = \lambda AC\,,\\
C_{xy} &= (A_y/A)C_x + (B_x/B)C_y\,, \quad (A_y/B)_y + (B_x/A)_x + A_zB_z/C^2 = \lambda AB\,.
\end{aligned}
$$

In the case $\lambda = 0$ we have the metric with zero curvature and the system determines a class of triple orthogonal coordinate system in Euclidean 3-dimensional space. Using matrix representation of the condition $R^i_{jkl} = 0$ one obtains:

$$
(9)\; \Gamma_{2,x} - \Gamma_{1,y} + [\Gamma_1, \Gamma_2] = 0\,, \quad \Gamma_{2,z} - \Gamma_{3,y} + [\Gamma_3, \Gamma_2] = 0\,, \quad \Gamma_{3,x} - \Gamma_{1,z} + [\Gamma_1, \Gamma_3] = 0\,,
$$

where Γ^i_{jk} are Christoffel symbols of the metric (7). We can write the system (11) as condition of compatibility of the following linear system of equations with spectral parameter q:

$$
\begin{aligned}
(q^3\partial/\partial_z + q\partial/\partial_x + \partial/\partial_y)\Phi &= -(q^3\Gamma_3 + q\Gamma_1 + \Gamma_2)\Phi\,,\\
(q^2\partial/\partial_y + q\partial/\partial_x + \partial/\partial_z)\Phi &= -(q^2\Gamma_2 + q\Gamma_1 + \Gamma_3)\Phi\,,
\end{aligned}
$$

It is known (see [1]) that this representation cannot be used for the integration of the Lame system. We can consider the system

$$
\begin{aligned}
(10)\qquad A_{zy} &= (C_y/C)A_z + (B_z/B)A_y\,,\\
B_{zx} &= (C_x/C)B_z + (A_z/A)B_x\,,\\
C_{xy} &= (A_y/A)C_x + (B_x/B)C_y\,,
\end{aligned}
$$

as independent subsystem of system (8). Of course this subsystem has no original geometrical meaning of the system (8). Nevertheless it has its own geometrical meaning in terms of the following construction. This subsystem can be considered as compatibility condition of the linear system of equations

$$
\begin{aligned}
(11)\qquad \Phi_{xy} &= (A_y/A)\Phi_x + (B_x/B)\Phi_y\,,\\
\Phi_{xz} &= (A_z/A)\Phi_x + (C_x/C)\Phi_z\,,\\
\Phi_{zy} &= (B_z/B)\Phi_y + (C_y/C)\Phi_z\,.
\end{aligned}
$$

Choosing five solutions $\Phi^i(x,y,z)$, $i - 0,\dots,4$, of the system (11) one can construct a 3-dimensional manifold $F(\Phi^4/\Phi^0, \Phi^3/\Phi^0, \Phi^2/\Phi^0, \Phi^1/\Phi^0) = 0$ in 4-dimensional projective space $\mathbb{R}P^4$. In this hyperspace there exist 2-dimensional surfaces which intersect on the thrice conjugate lines. This follows from the projective generalization of the Dupin theorem of classical differential geometry on a family of curvature lines. The systems (10) and (11) were studied earlier [1] as examples of the Zakharov – Manakov [2] multidimensional integrable equations. The system (10) can be represented in the operator form:

$$
\begin{aligned}
[L_1, L_2] &= (B_{xz} - A_{xz})L_1 + (A_{xy} - C_{xy})L_2 + (B_{xx} - C_{xx})L_3\,,\\
[L_1, L_3] &= (A_{zy} - B_{zy})L_1 + (A_{yy} - C_{yy})L_2 + (B_{xy} - C_{xy})L_3\,,\\
[L_2, L_3] &= (A_{zz} - B_{zz})L_1 + (A_{zy} - C_{zy})L_2 + (C_{xz} - B_{xz})L_3\,,
\end{aligned}
$$

where the operators L_i have the form:

$$L_1 = D_{xy} - (A_y/A)D_x - (B_x/B)D_y \,,$$
$$L_2 = D_{xz} - (A_z/A)D_x - (C_x/C)D_z \,,$$
$$L_3 = D_{yz} - (B_z/B)D_y - (C_y/C)D_y \,.$$

We can construct solution of the systems (10), (11) using representation:

$$\Phi = \left[\lambda^n + a_1\lambda^{n-1} + \ldots + a_{n-1}\lambda + a_n\right] \exp\left[\frac{x}{\lambda} + \frac{y}{\lambda-1} + \frac{z}{\lambda+1}\right] , \tag{12}$$

where $a_i = a_i(x, y, z)$, λ is a parameter. So, for $\mathbf{n = 0}$ after substitution of Φ in (12) one obtains the system for $S = 1/a_0$:

$$S_{xy} = S_y - S_x \,, \quad S_{zx} = S_x - S_z \,, \quad S_{zy} = S_y/2 - S_z/2 \,. \tag{13}$$

Thus we have:

$$A = \frac{e^{z-y}}{S} \,, \qquad B = \frac{e^{x+z/2}}{S} \,, \qquad C = \frac{e^{-x-y/2}}{S} \,. \tag{14}$$

$\mathbf{n = 1}$. In this case we have the system of equations for $a(x, y, z) \equiv a_1(x, y, z)$:

$$a_{xy} = (a_y + a_x a_y)/a + (a_x + a_y a_x)/(a+1) + a_y - a_x \,,$$
$$a_{xz} = (a_z + a_x a_z)/a + (a_x + a_x a_z)/(a-1) + a_x - a_z \,,$$
$$a_{yz} = (a_z + a_y a_z)/(a+1) + (a_y + a_y a_z)/(a-1) + a_y/2 - a_z/2 \,.$$

This system is equivalent to the following system of linear equations:

$$S_{yz} = -(k/2)S + (k - 1/2)S_y + (1/2)S_z \,,$$
$$S_{xy} = -S \,, \qquad S_{xz} = (2k-1)S + kS_x - 2S_z \,,$$

where $a = S_y/(S - S_y)$, and k is a parameter. We can receive the solutions of initial equations (10), based on the solutions of the system (13). So we have:

$$A = ae^{z-y} \,, \qquad B = (a+1)e^{x+z/2} \,, \qquad C = (a-1)e^{-x-y/2} \,. \tag{15}$$

More general classes of solutions are obtained in [3]. A natural 4–dimensional generalization of the system (10) may be obtained in the analogous way for the metric:

$$ds^2 = \Phi^2 dt^2 + A^2 dx^2 + B^2 dy^2 + C^2 dz^2 \,. \tag{16}$$

Then the condition (1) for $i = k$ leads to equations, which are compatibility conditions of the following system:

$$\Phi_{xy} = (A_y/A)\Phi_x + (B_x/B)\Phi_y, \;\; \Phi_{xz} = (A_z/A)\Phi_x + (C_x/C)\Phi_z \,,$$
$$\Phi_{zy} = (B_z/B)\Phi_y + (C_y/C)\Phi_z, \;\; \Phi_{yt} = (B_t/B)\Phi_y + (D_y/D)\Phi_t \,,$$
$$\Phi_{tz} = (D_z/D)\Phi_t + (C_t/C)\Phi_z, \;\; \Phi_{zy} = (B_z/B)\Phi_y + (C_y/C)\Phi_z \,.$$

These equations give 4–dimensional generalization of equation (10) at $\lambda = 0$.Using representation of the type (12) we can obtain the solution of this system. In the simplest case:

$$\Phi = a(x,y,z,t)\exp\left[\lambda t + \frac{x}{\lambda} + \frac{y}{\lambda-1} + \frac{z}{\lambda+1}\right] \tag{17}$$

and $\Phi = a(x,y,z,t)$, $A = a(x,y,z,t)e^{z-y}$, $B = a(x,y,z,t)e^{x+t+z/2}$, $C = a(x,y,z,t)e^{y/2-x-t}$, where function $a(x,y,z,t)$ is determined by means of solutions of a corresponding linear system of equations.

3. EINSTEIN EQUATION FOR THE STATIONARY GRAVITATIONAL FIELDS

Let us consider the condition $R^l_{ilk} = R_{ik} = 0$ for the metric:

$$ds^2 = \Phi^2(x,y,z)dt^2 - A^2(x,y,z)dx^2 - B^2(x,y,z)dy^2 - C^2(x,y,z)dz^2\,. \tag{18}$$

Then one obtains the system of equations:

$$\begin{aligned}
\Phi_{xy} &= (A_y/A)\Phi_x + (B_x/B)\Phi_y - [C_{xy} - (A_y/A)C_x - (B_x/B)C_y]\Phi/C\,,\\
\Phi_{xz} &= (A_z/A)\Phi_x + (C_x/C)\Phi_z - [B_{zx} - (C_x/C)B_z - (A_z/A)B_x]\Phi/B\,,\\
\Phi_{zy} &= (B_z/B)\Phi_y + (C_y/C)\Phi_z - [A_{zy} - (C_y/C)A_z - (B_z/B)A_y]\Phi/A\,,\\
\Phi_{xx} &= A_x\Phi_x/A - AA_y\Phi_y/B^2 - AA_z\Phi_z/C^2 + A^2E_1\Phi\,,\\
\Phi_{yy} &= B_y\Phi_y/B - BB_x\Phi_x/A^2 - BB_z\Phi_z/C^2 + B^2E_2\Phi\,,\\
\Phi_{zz} &= C_z\Phi_z/C - CC_x\Phi_x/A^2 - CC_y\Phi_y/B^2 + C^2E_3\Phi\,,
\end{aligned} \tag{19}$$

where

$$\begin{aligned}
E_1 &= [(B_z/C)_z + (C_y/B)_y + B_xC_x/A^2]/(BC)\,,\\
E_2 &= [(C_x/A)_x + (A_z/C)_z + A_yC_y/B^2]/(AC)\,,\\
E_3 &= [(A_y/B)_y + (B_x/A)_x + A_zB_z/C^2]/(AB)\,,
\end{aligned}$$

which must obey the main condition:

$$E_1 + E_2 + E_3 = 0\,. \tag{20}$$

Thus the solution of this system of equations determines all types of the stationary gravitational fields.

4. EISENHART SPACES AND THEIR APPLICATIONS

Now we consider the class of metrics (7) with condition ${}^3R = 0$. Due to (3) for components of metric tensor this condition is equivalent to the equation (20). This relation takes remarkable form under ansatz:

$$A = B = e^{S(x,y,z)}\,, \qquad C = S_z\,. \tag{21}$$

After this substitution in the relation (20) one obtains the equation

$$[S_{xx} + S_{yy}]_z + [S_{xx} + S_{yy}]S_z + 3S_z e^{2S} = 0\,, \tag{22}$$

which after integration in z can be written as

$$S_{xx} + S_{yy} = A(x,y)e^{-S} - e^{2S}\,, \tag{23}$$

with arbitrary function $A(x,y)$. Choosing here $A = 1$ one obtains the equation which is very well known in the theory of the Inverse scattering transform method. We can use this method for integration of the equation (23). Let us consider any solution of (23) depending on some parameters b_k. Choosing variable z to be one of these parameters we get solution of equation (22): $(S_{xx} + S_{yy})_{b_k} + S_{b_k}(S_{xx} + S_{yy}) + 3S_{b_k}e^{2S} = 0$. Other integrable cases corresponds to the condition $A(x,y) = 0$, when (23) is reduced to the Liouville equation.

Condition ${}^3R = 0$ has remarkable applications in theory of gravitational waves [4]. In fact, splitting of the full system of Einstein equations in two parts leads to the following conditions on initial data:

$${}^3R + (\mathrm{Sp}\,K)^2 - \mathrm{Sp}(K^2) = 0\,, \quad (K^i_j - \delta^i_j\,\mathrm{Sp}\,K)_{,i} = 0\,, \tag{24}$$

where K^i_j are some functions constructed from the components of metric tensor

$$ds^2 = {}^3g_{ik}dx^i dx^k + 2N_i dt dx^i + (N_i N^j - N^2)dt^2\,. \tag{25}$$

The case ${}^3R = 0$ (when $K^i_j = 0$) correspondence to the configurations of gravitational waves symmetric in time. Notice, that we can generalize this approach on the initial data in gravitational theory using the integrable system:

$$S_{xt} = e^{2S} - \cosh[3Q]e^{-S}\,, \quad Q_{xt} = \sinh[3Q]e^{-S}\,, \tag{26}$$

which coincides with (23) for $Q = 0$. N-parametric solutions of this system can be used for investigation the full system (24).

5. YANG-MILLS CONDITIONS ON CURVATURE

Let Γ^k_{ij} be the connection coefficients of a manifold. Yang-Mills condition on curvature tensor has the form (4), where $R^j_{ilk} = \Gamma^j_{il,k} - \Gamma^j_{lk,j} + [\Gamma_i, \Gamma_k]^j_l$ is the Riemannian tensor of curvature.

It can be shown that matrixes R_{ij} obeying conditions

$$R_{12} + R_{30} = 0\,, \qquad R_{32} + R_{01} = 0\,, \qquad R_{13} = R_{02} = 0 \tag{27}$$

resolve the Yang-Mills equation on curvature. These conditions give us nonlinear system of equations on Γ_i which is special reduction of the full system:

$$\begin{aligned}
&\Gamma_{1y} - \Gamma_{2x} + [\Gamma_1, \Gamma_2] + \Gamma_{3t} - \Gamma_{0z} + [\Gamma_3, \Gamma_0] = 0\,,\\
&\Gamma_{3y} - \Gamma_{2z} + [\Gamma_3, \Gamma_2] + \Gamma_{0x} - \Gamma_{1t} + [\Gamma_0, \Gamma_1] = 0\,,\\
&\Gamma_{0y} - \Gamma_{2t} + [\Gamma_0, \Gamma_2] = 0, \quad \Gamma_{1z} - \Gamma_{3x} + [\Gamma_1, \Gamma_3] = 0\,.
\end{aligned}$$

Notice that this reduction of the Yang-Mills equation differs from the well known self-dual one. We can represent this system as condition of compatibility of the following linear system of equations:

$$\lambda^2\Phi_y + \lambda\Phi_t + \Phi_x = (\lambda^2\Gamma_2 + \lambda\Gamma_0 + \Gamma_1)\Phi\,,$$
$$\lambda^2\Phi_t - \lambda\Phi_y + \Phi_z = (\lambda^2\Gamma_0 - \lambda\Gamma_2 + \Gamma_3)\Phi\,.$$

Using standard technique of the Inverse Scattering Transform Method one can construct solutions of these equations. In the simplest case $\Gamma_0 = M_{zx}$, $\Gamma_2 = -M_{zz}$, $\Gamma_1 = \Gamma_3 = 0$ one obtains the matrix system

$$M_{zzt} + M_{zxy} + [M_{zz}, M_{zx}] = 0\,, \quad M_{zzz} + M_{zxx} = 0\,, \tag{28}$$

which can be examined in the elementary way.

6. GEOMETRICAL THEORY OF THE SECOND ORDER EQUATIONS

Cartan theory [5] of the second order differential equations (5) is based on the following system of differential 1-form determining in the space of linear elements (x, y, y') some projective connection. These forms have components:

$$w^1 = dx\,, \quad w^2 = dy - y'dx\,, \quad w_1^2 = dy' - f(x,y,y')dx\,,$$
$$w_1^1 - w_0^0 = f_{y'}dx\,, \quad w_2^2 - w_0^0 = -f_{y'y'}(dy - y'dx)/2\,,$$
$$w_2^1 = f_{y'y'y'}(dy - y'dx)/6\,,$$
$$w_1^0 = f_y dx + f_{y'y'}w_1^2/2 + [2f_{yy'}/3 - (1/6)d/dx(f_{y'y'}]w^2\,,$$
$$w_2^0 = [f_{yy'}/3 - (1/6)d/dx(f_{y'y'})]dx - d(f_{y'y'})/6 + mw^2\,,$$

where $6m = f_{yy'y'} - f_{y'}f_{y'y'y'} - (d/dx)f_{y'y'y'}$, and $d/dx = \partial/\partial x + y'\partial/\partial y + f\partial/\partial y'$. The curvature tensor of the connection has the components:

$$R_2^1 = aw^2 \wedge w_1^2\,, \quad R_1^0 = bw^1 \wedge w^2\,, \quad R_2^0 = hw^1 \wedge w^2 + kw^2 \wedge w_1^2\,, \tag{29}$$

where $6a = -f_{y'y'y'y'}$, $6k = -6m_{y'} - f_{y'y'}f_{y'y'y'}$, $h = b_{y'}$ and $6b = -f_{xxy'y'} - 2y'f_{xyy'y'} - y'^2 f_{yyy'y'} - 2y'ff_{yy'y'y'} - 2ff_{xy'y'y'} - f^2 f_{y'y'y'y'} - (f_x + y'f_y)f_{y'y'y'} + 4f_{xyy'} + 4y'f_{yyy'} + (3f + y'f_{y'})f_{yy'y'} + f_{y'}f_{xy'y'} + 4f_{y'}f_{yy'} + 3f_y f_{y'y'} - 6f_{yy}$. Using the connection and curvature Cartan constructed invariants of equation under transformations $x = U(x,y)$, $y = V(x,y)$ (this problem was solved earlier by T. Tresse without geometrical interpretation). More earlier Liouville [6] has constructed some series of invariants for the simplest form–invariant equations:

$$y'' + a_1(x,y)y'^3 + a_2(x,y)y'^2 + a_3(x,y)y' + a_4(x,y) = 0\,. \tag{30}$$

In this case one obtains the connection in 2–dimensional space (x, y):

$$\Pi_{11} = 2(a_3^2 - a_2a_4) + a_{3x} - a_{4y}\,, \quad \Pi_{22} = 2(a_2^2 - a_1a_3) + a_{1x} - a_{2y}\,,$$
$$\Pi_{12} = \Pi_{21} = (a_2a_3 - a_1a_4) + a_{2x} - a_{3y}\,,$$

and the components of curvature tensor:

$$\begin{aligned} L_1 &= \Pi_{12x} - \Pi_{11y} + 2a_3\Pi_{12} - a_4\Pi_{22} - a_2\Pi_{11}\,, \\ L_2 &= \Pi_{22x} - \Pi_{12y} + 2a_2\Pi_{12} - a_3\Pi_{22} - a_1\Pi_{11}\,. \end{aligned} \tag{31}$$

Using L_1, L_2 and $a_i(x,y)$ it is possible to obtain the series of invariants:

$$\begin{aligned} N_{i+2} &= L_1N_{iy} - L_2N_{ix} + iN_i(L_{2x} - L_{1y})\,, \quad i > 5\,, \\ N_5 &= L_2(L_1L_{2x} - L_2L_{1x}) + L_1(L_2L_{1y} - L_1L_{2y}) - \\ &\quad - a_1L_1^3 + 3a_2L_1^2L_2 + a_4L_2^3 - 3a_3L_2^2L_1\,. \end{aligned} \tag{32}$$

In the theory of the second order equations some invariants has geometrical sense. The fundamental role of invariant N_5 for equation (30) was found in [5].

Theorem 1. *The space of projective connection, determined by the equation* (30) *under condition* $N_5 = 0$ *can be immersed as developable surface in the projective space* $\mathbb{R}P^3$*, and in space* $\mathbb{R}P^4$ *if* $N_5 \neq 0$ [7].

According to this theorem all equations of the type (30) can be divided in two classes, and theory of surfaces in projective spaces can be used in investigation of these equations. We can use the theory of invariants for the studying of some nonlinear dynamical system. For example, the Lorentz system of equations (6) is equivalent to the one second order differential equation

$$y'' = (y')^2 3/y + (1/x - my)y' + (nx^3 - lx)y^4 + (nx^2 + t)y^3 - sy^2/x\,, \tag{33}$$

where

$$m = (b + k + 1)/k, \quad n = 1/k^2, \quad t = b(k+1)/k^2, \quad s = (k+1)/k, \quad l = b(r-1)/k^2. \tag{34}$$

In this case we have $a_1 = 0$, $a_2 = -1/y$, $3a_3 = (my - 1/x)$, $a_4 = (lx - nx^3)y^4 - (nx^2 + t)y^3 + sy^2/x$, and the components of curvature tensor (31):

$$L_1 = 3(lx - nx^3)y^2 - 2m^2y/3 + (2m/3 - s)/x\,, \qquad L_2 = m/y\,. \tag{35}$$

According to (32) we have the expression for semiinvariant:

$$\begin{aligned} N_5 &= mn(10m - 6s - m^2)x^2 + m(4m^2/9 + 2ms/3 - 2s^2)/x^2y^2 + \\ &\quad + m(2m^4/9 + 6ls - 4ml - m^2t)\,. \end{aligned}$$

Due to the theorem one obtains the following result. Invariant N_5 is identically equal to zero only in two cases: **1)** $s = -1/5$, $b = -16/5$, $r = -7/5$; **2)** $s = 1/3$, $b = 2/3$, r is arbitrary. Thus for these choices of parameters equation (33) determines two–dimensional surface of projective connection immersed in $\mathbb{R}P^3$–space. Other values of parameters lead to the surfaces in the space $\mathbb{R}P^4$.

Note. In $\mathbb{R}P^3$ some surfaces are associated with integrable nonlinear differential equations. For example the system of equations

$$\Phi_{xy} = a(x,y)\Phi_x + b(x,y)\Phi\,, \qquad \Phi_{xx} = q\Phi_y + 2qa(x,y)\Phi\,, \tag{36}$$

where q is a parameter, and for which integrable condition of compatibility has the form

$$3q^2 a_{yy} - 6q(a^2)_{xx} + a_{xxxx} = 0 \tag{37}$$

determines some developable surface $F(\Phi^1/\Phi^0, \Phi^2/\Phi^0, \Phi^1/\Phi^0) = 0$ in $\mathbb{R}P^3$–space. Here $\Phi^i(x,y)$ are the solutions of the linear system for some $a(x,y)$. The examples of the surfaces in $\mathbb{R}P^4$ can be constructed by means of the solutions of the linear system of equations (11).

In the theory of the second order differential equations (5) relation

$$F(x, y, a, b) = 0 \tag{38}$$

presenting the general integral of the equation is of great importance. By its means dual equation is introduced:

$$b'' = \varphi(a, b, b'). \tag{39}$$

Proposition 1. *For the equations of the form* (30) *dual equations are determined by the condition* $(c = b')$:

$$\begin{aligned}&\varphi_{aacc} + 2c\varphi_{abcc} + 2\varphi\varphi_{accc} + c^2\varphi_{bbcc} + 2c\varphi\varphi_{bccc} + \varphi^2\varphi_{cccc} + (\varphi_a + c\varphi_b)\varphi_{ccc} - \\ &\quad -4\varphi_{abc} - 4c\varphi_{bbc} - 3\varphi\varphi_{bcc} - c\varphi_c\varphi_{bcc} - \varphi_c\varphi_{acc} + 4\varphi_c\varphi_{bc} - 3\varphi_b\varphi_{cc} + 6\varphi_{bb} = 0\,.\end{aligned}$$

This equation has solutions of the form [8]:

$$\varphi = c^k U[ac^{k-1}], \qquad \varphi = c^k U[bc^{k-2}], \qquad \varphi = c^k U[ac^{k-1}, bc^{k-2}]. \tag{40}$$

In the case $\varphi = a^{-k}A(ca^{k-1})$ one obtains $(x = ca^{k-1})$:

$$\begin{aligned}&[A + (k-1)x]^2 A^{IV} + 3(k-2)[A + (k-1)x]A^{III} + \\ &\quad +(2-k)A^I A^{II} + (k^2 - 5k + 6)A^{II} = 0\,.\end{aligned}$$

Solution of this equation is given by

$$A = (2-k)[x(1+x^2) + (1+x^2)^{3/2}] + (1-k)x\,, \tag{41}$$

where k is a parameter. For $k = 1$ we have equation $ab'' = b'(1+b'^2) + (1+b'^2)^{3/2}$, which is dual to the equation $2xy'' = -y'^3 - y'$. The solutions of dual equation determines the structure of the splitting of the phase space in trajectories and is associated with its Finsler properties in the every domain.

Other way of introducing of the Finsler metric in theory of the ordinary differential equations is connected with their Lagrangian form. For the second order equation (5) we have the condition for Lagrangian $ds = A(x,y,y')dx$: $y'A_{yy'} + A_{xy'} + f(x,y,y')A_{y'y'} - A_y = 0$, which can be written in the form $y'S_y + S_x + f(x,y,y')S_{y'} + f_{y'} = 0$, where $S = \log A_{y'y'}$.

Proposition 2. *The condition*

$$\det \begin{vmatrix} S' & 1 & 0 & 0 & P \\ S'' & S' & 1 & 0 & P' \\ S''' & 2S'' & S' & 1 & P'' \\ S^{IV} & 3S''' & 3S'' & S' & P''' \\ S^V & 4S^{IV} & 6S''' & 4S'' & P^{IV} \end{vmatrix} = 0\,, \tag{42}$$

where $P = -y'S_y - S_x$, $S' = S_{y'}, \ldots, P' = P_{y'}, \ldots$ *determines all invariant Finsler metrics for equations* (30).

The following result for the equations (33), (34) is obtained: Lorentz equation can be considered as the equation of geodesic lines in a space with the Finsler metric $ds = [a(x,y)y' + b(x,y)]^{1/n} dx$ in the case $k = n/(4-n)$, $b = 8(n-1)/(4-n)$, $r = (3n-4)/(4-n)$; and $ds = [a(x,y)y' + b(x,y)]^{1-n}(y')^{n+1}$ in the case $k = 1/(4n+1)$, $b = 8/(4n+1)$, $r = (16n+5)/(4n+1)$ or $k = 1/(2n+1)$, $b = 2/(2n+1)$, r is arbitrary. Here $a(x,y)$, $b(x,y)$ are some functions. In the work [9] it was shown that this class of metrics corresponds to the Berwald spaces. Notice, that appearance of the Finsler properties in the theory of the second order equations is based on the projective connection in the space (x, y, y').

There exist a remarkable correspondence of invariant N_5, Painlevé property and the Finsler metrics for equation (33). It is known that Painlevé property (or complete integrability) is fulfilled for the following choices of parameters: **1)** $s = 0$, **2)** $s = 1/2, b = 0, r = 0$, **3)** $s = 1$, $b = 2$, $r = 1/9$, **4)** $s = 1/3$, $b = 0$, r is arbitrary. Comparing these conditions with the conditions of existence of Finsler metric for the equation (33) we discover that some of them coincide. For example, we have the following relations for parameters: $A = 0$ (i.e. $3s^2 + 8bs + 2s = (b+1)^2$), $B = 0$ (i.e. $s = 2b - 1$ and $4s + b + 4 = 0$), $C = 0$ (i.e. $r = 1 + [(b+s+1)^2(b-2s-2)/18bs]$). From the last condition we have: $s = 1$, $b = 2$, $r = 1/9$, i.e. the case **3** above.

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ГЕОМЕТРИЧЕСКИЕ СВОЙСТВА МНОГОМЕРНЫХ НЕЛИНЕЙНЫХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ И ФИНСЛЕРОВСКИЕ МЕТРИКИ ФАЗОВЫХ ПРОСТРАНСТВ ДИНАМИЧЕСКИХ СИСТЕМ

Рассмотрены многомерные нелинейные дифференциальные уравнения, возникающие из специфических условий на тензор кривизны трех- и четырехмерных многообразий. Построены примеры финслеровских метрик, связанных с нелинейными динамическими системами.